T = absolute temperature
u = internal energy ($\Delta u^o = u - u_{ideal}$)
z = compressibility factor ($z = pv/RT$)

Greek Letters

Δ = temperature difference between argon and selected gas at z minimum
ϵ/k = empirical reduction constant
ρ = density
$\rho_{c'}$ = pseudo-critical density ($\rho_{c'} = p_c/RT_c$)
μ^o = Joule-Thomson coefficient at zero pressure

Superscript

$*$ = reduced or generalized values

LITERATURE CITED

1. Hirschfelder, J. O., C. F. Curtiss, and R. B. Bird, "Molecular Theory of Gases and Liquids," Wiley, New York (1954).
2. Barker, J. A., and J. J. Monaghan, *J. Chem. Phys.*, **36**, 2564 (1962).
3. Boys, S. F., and I. Shavitt, *Proc. Roy. Soc.*, **A254**, 487 (1960).
4. Nelson, L. C., and E. F. Obert, *Trans. Am. Soc. Mech. Engrs.*, **76**, 1057 (1954).
5. ——————, *A.I.Ch.E. Journal*, **1**, p. 74 (March, 1955).
6. Gyorog, D. A., and E. F. Obert, *A.I.Ch.E. Journal*, **10**, 5 (Sept., 1964).
7. Nelson, L. C., Ph. D. thesis, Northwestern Univ., Evanston, Illinois (1954).
8. Michels, A. H., J. M. Levelt, and W. de Graaff, *Physica*, **24**, 659 (1958).
9. ——————, and G. J. Wolkers, *ibid.*, p. 769.
10. ———, H. B. Wijker, and H. K. Wijker, *ibid.*, **15**, 627 (1949).
11. ———, R. J. Lunbeck, and G. J. Wolkers, *ibid.*, p. 689.
12. Din, F., "The Thermodynamic Functions of Gases—Air Propane, Acetylene, and Argon," Vol. 2, Butterworths, London, England (1959).
13. Bird, R. B., Ph. D. thesis, Univ. Wisc., Madison, Wisconsin (1950).
14. Sage, B. H., and W. N. Lacey, "Thermodynamic Properties of the Lighter Paraffin Hydrocarbons and Nitrogen," Monograph on API Res. Proj. 37, Am. Pet. Inst. (1950).
15. Din, F., ed., "The Thermodynamic Functions of Gases—Ethane, Methane, and Nitrogen," Vol. 3, Butterworths, London, England (1961).
16. Michels, A. H., G. J. Wouters, and J. de Boer, *Physica*, **3**, 585 (1936).
17. *Ibid.*, p. 597.
18. Michels, A. H., R. J. Lunbeck, and G. J. Wouters, *ibid.*, **17**, 801 (1951).
19. Gyorog, D. A., Ph. D. thesis, Univ. Wisc., Madison, Wisconsin (1963).
20. Michels, A. H., J. M. Lupton, T. Wassenaar, and W. de Graaff, *Physica*, **18**, 121 (1952).
21. Ishkin, I. P., and M. G. Kaganer, *Soviet Phys.-Tech. Phys.*, **1**, 2255 (1956).
22. Kihara, T., *Rev. Mod. Phys.*, **25**, 831 (1953).
23. Obert, E. F., "Concepts of Thermodynamics," McGraw-Hill, New York (1960).
24. Benedict, M., G. B. Webb, and L. C. Rubin, *J. Chem. Phys.*, **3**, 334 (1940).
25. Van Itterbeek, W. de Rop, and G. Forrez, *Appl. Sci. Res. Sect. A*, **6**, 421 (1957).

Manuscript received August 26, 1963; revision received December 5, 1963; paper accepted December 9, 1963.

Velocity Distributions in Two-Dimensional Laminar Liquid-into-Liquid Jets in Power-Law Fluids

CHAIM GUTFINGER and REUEL SHINNAR

Department of Chemical Engineering Technion, Israel Institute of Technology, Haifa, Israel

The study of boundary-layer equations for non-Newtonian fluids has lately become of interest. It has been shown (*1, 4, 7, 11, 14, 15*) that these equations become mathematically treatable (*15*) for the simplified case of the so-called *power-law* fluid. Numerical solutions have been published for a boundary layer near a flat plate (*1*). In this paper the flow behavior of a two-dimensional laminar liquid-into-liquid jet will be investigated, and complete analytical solutions will be presented.

The case of the two-dimensional jet is important in the study of the laminar mixing of non-Newtonian fluids as it represents a simple model of the entrainment process. Furthermore the fact that the solution is analytical is of special value in investigating the general properties of boundary-layer equations in non-Newtonian fluids.

Chaim Gutfinger is at Yale University, New Haven, Connecticut. Reuel Shinnar is at City University of New York, New York, New York.

THE BOUNDARY-LAYER EQUATIONS FOR A NON-NEWTONIAN FLUID

The general equation for steady state two-dimensional flow of an incompressible liquid can be written as

$$\rho u_j \frac{\partial u_i}{\partial x_j} = \frac{\partial \tau_{ij}}{\partial x_j} \tag{1}$$

Rivlin and Ericksen (*9, 10*) have shown that for any isotropic incompressible fluid with no memory the stress tensor τ_{ij} can be defined as a polynomial function of the kinematic matrices. In cases where only the first- and second-order matrices are required to define τ_{ij}

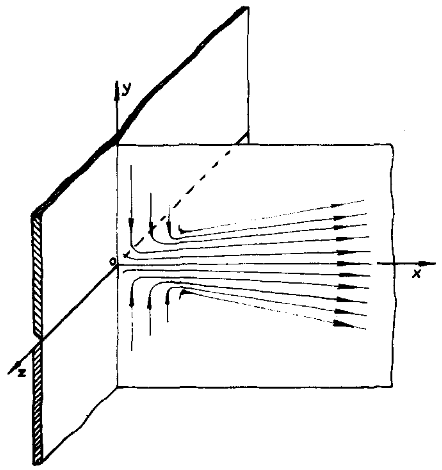

Fig. 1. Schematic representation of a two-dimensional jet.

$$\tau_{ij} = \alpha_0\delta_{ij} + \alpha_1 D_1 + \alpha_2 D_2 + \alpha_3 D_1^2 + \alpha_4 D_2^2 + \alpha_5 (D_1 D_2 + D_2 D_1) + \alpha_6 (D_1^2 D_2 + D_2 D_1^2) + \alpha_7 (D_1 D_2^2 + D_2^2 D_1) + \alpha_8 (D_1^2 D_2^2 + D_2^2 D_1^2) \qquad (2)$$

where τ_{ij} is the stress matrix, and D_1 and D_2 are the kinematic matrices of first and second order respectively defined by

$$D_1 = \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \qquad (2a)$$

$$D_2 = \frac{D}{Dt}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \frac{\partial u_m}{\partial x_i}\left(\frac{\partial u_m}{\partial x_j} + \frac{\partial u_j}{\partial x_m}\right) + \frac{\partial u_m}{\partial x_j}\left(\frac{\partial u_i}{\partial x_m} + \frac{\partial u_m}{\partial x_i}\right) \qquad (2b)$$

The coefficients $\alpha_1, \alpha_2 \ldots\ldots \alpha_8$ are functions of the invariants of the matrices in Equation (2).

A full description of a non-Newtonian fluid in two-dimensional flow can be given only by including all the terms in Equation (2). However the coefficients $\alpha_1 \ldots\ldots \alpha_8$ in Equation (2) are difficult to evaluate experimentally. It has therefore been found useful to study the simplified case of a fluid where all coefficients but α_1 are assumed to be zero. It is further assumed that the function α_1 can be approximated by

$$\alpha_1 = m\left\{ a_o \left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2 \right]\right\}^{\frac{n-1}{2}} \qquad (3)$$

For the majority of non-Newtonian fluids the above mentioned simplifications are not generally valid. It has been shown experimentally that for most fluids exhibiting non-Newtonian behavior at least some of the constants $\alpha_2 \ldots \alpha_n$ are of appreciable magnitude. However these simplifications have proven to be useful in the study of several practical flow problems especially in boundary-layer flow. In this case these simplifications can be interpreted from a more general point of view. They simply are an extension of the boundary-layer assumptions usually made for Newtonian fluids. All terms containing the coefficients $\alpha_2 \ldots\ldots \alpha_8$ are assumed to be very small as compared with the terms retained in the boundary-layer equations, though the coefficients themselves may not be negligible. The limitations of such boundary-layer assumptions for a general fluid are discussed by Shinnar (*14*) in a separate paper. It should be mentioned that in some special cases boundary-layer solutions can also be obtained, including higher order coefficients (*4, 7*), but a discussion of this work is outside the scope of this paper.

For the purpose of this investigation it is assumed that the fluid can be fully described by Equation (3). Inserting Equation (3) into (1) one obtains

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \frac{1}{\rho}\frac{\partial}{\partial x}\left[\alpha_1\left(2\frac{\partial u}{\partial y}\right)\right] + \frac{1}{\rho}\frac{\partial}{\partial y}\left[\alpha_1\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right] \qquad (4)$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial u}{\partial x} = -\frac{1}{\rho}\frac{\partial P}{\partial y} + \frac{1}{\rho}\frac{\partial}{\partial x}\left[\alpha_1\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right] + \frac{1}{\rho}\frac{\partial}{\partial y}\left[\alpha_1\left(2\frac{\partial u}{\partial y}\right)\right]$$

α_1 is given by Equation (3). The order of magnitude of each term in the above equation can now be evaluated by applying the classical boundary-layer assumptions. The boundary-layer equation obtained this way is

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \frac{m}{\rho}\frac{\partial}{\partial y}\left\{\left[\left(a_o\frac{\partial u}{\partial y}\right)^2\right]^{\frac{n-1}{2}}\frac{\partial u}{\partial y}\right\} \qquad (5)$$

This can be written as

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{1}{\rho}\frac{\partial P}{\partial x} + B\epsilon\frac{\partial}{\partial y}\left|\frac{\partial u}{\partial y}\right|^n \qquad (6)$$

where

$$\epsilon = \frac{\partial u}{\partial y}\bigg/\left|\frac{\partial u}{\partial y}\right| \quad \text{and} \quad B = \frac{m}{\rho}a_o^{\frac{n-1}{2}}$$

This is the general form of the two-dimensional boundary-layer equation for a power-law fluid.

THE TWO-DIMENSIONAL LAMINAR JET

The two-dimensional laminar jet can be described as follows (Figure 1). A stream of liquid is emerging from an infinitely long, narrow slit, in an infinite plane, into an infinite volume of the same liquid.

As the slit is assumed to be infinitely narrow, an infinite velocity has to be postulated at the origin of the jet. To define the problem it is further assumed that the entering fluid possesses a finite momentum (M) in the x direction.

For a very narrow stream the flow equation can be simplified to Equation (6) by the application of the boundary-layer assumptions given above. The pressure term in Equation (6) can be neglected as the stagnant fluid impresses its pressure on the thin flowing sheet (*12*). Therefore

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \epsilon B\frac{\partial}{\partial y}\left|\frac{\partial u}{\partial y}\right|^n \qquad (7)$$

with the boundary conditions

$$y = 0\ ;\ \frac{\partial u}{\partial y} = 0,\ v = 0$$

$$y = \infty;\ u = 0$$

An additional condition can be derived from the fact that at constant surrounding pressure the total momentum (M) of the fluid in the x direction must be preserved.

This can be stated as follows:

$$K = \frac{M}{\rho} = \int_{-\infty}^{\infty} u^2 dy = \text{const} \tag{8}$$

A complete analytical solution of this case has been obtained by Gutfinger (3) and Kapur (5),* a summary of which is brought here. A stream function defined by

$$u = \frac{\partial \psi}{\partial y}; \quad v = -\frac{\partial \psi}{\partial x} \tag{9}$$

satisfies (7) and (8) by

$$\psi = [3nB\, \beta^{2n-1}]^{\frac{1}{2-n}}\, x^{\frac{1}{3n}}\, F(\xi) \tag{10}$$

$$\xi = \beta x^{-2/3n}\, y \tag{11}$$

thus defining u and v as

$$u = [3nB\, \beta^{n+1}]^{\frac{1}{2-n}}\, x^{-1/3n}\, F'(\xi) \tag{12}$$

$$v = \frac{1}{3n}[3nB\, \beta^{2n-1}]^{\frac{1}{2-n}}\, x^{\frac{1}{3n}-1}\, [2\,\xi\, F' - F] \tag{13}$$

Equation (7) is transformed into

$$F'^2 + FF'' = -\epsilon \frac{d}{d\xi}|F''|^n \tag{14}$$

with the boundary conditions

$$\xi = 0\ ;\ F = 0,\ F'' = 0$$
$$\xi = \infty;\ F' = 0$$

The sign constant ϵ is now defined as $\epsilon = \frac{F''}{|F''|}$.

Integration of Equation (14) yields

$$F' = \left[\ 1 + \frac{s-2}{s+1}|F|^{s+1}\right]^{\frac{1}{2-s}} \qquad s \neq 2$$

$$F' = e^{-1/3\, F^3} \qquad s = 2 \tag{15}$$

where $s = 1/n$. F' has been normalized to give $F' = 1$ at $\xi = 0$. From this ξ can be extracted to yield

$$\xi = \int \left[1 + \frac{s-2}{s+1}|F|^{s+1}\right]^{\frac{1}{s-2}} dF, \quad s \neq 2$$

$$\xi = \int \exp\frac{|F|^3}{3}\, dF \qquad s = 2 \tag{16}$$

This integral may be evaluated in closed form for some values of s which leads to a complete analytical solution of the problem. For any value of s this integral can be easily evaluated to any desired degree of accuracy by numerical methods. Some examples are given in the Appendix.

In order to obtain a complete solution for u and v it remains only to evaluate the free constant β which can be obtained from the condition of constant momentum in the x direction:

$$K = (3nB)^{\frac{1}{2-n}}\, \beta^{\frac{3n}{2-n}} \int_{-\infty}^{\infty} F'^2\, d\xi = \text{const} \tag{17}$$

β can be evaluated from Equation (17) in terms of the integral A

$$A = \int_{-\infty}^{\infty} F'^2\, d\xi = \int_{F(-\infty)}^{F(\infty)} F' dF \tag{18}$$

which is a constant for a given n. Substitution of F' from (15) gives an integral that can be reduced to Euler's integral of the first kind and solved to yield

$$A = 2\left(\frac{s+1}{2-s}\right)^{\frac{1}{s+1}} \frac{\Gamma\left(\frac{s+2}{s+1}\right)\Gamma\left(\frac{3-s}{2-s}\right)}{\Gamma\left(\frac{1}{s+1} + \frac{3-s}{2-s}\right)} \qquad \begin{matrix} s < 2 \\ n > 1/2 \end{matrix}$$

$$A = 2\left(\frac{s+1}{s-2}\right)^{\frac{1}{s+1}} \frac{\Gamma\left(\frac{s+2}{s+1}\right)\Gamma\left(\frac{1}{s-2} - \frac{1}{s+1}\right)}{\Gamma\left(\frac{1}{s-2}\right)} \qquad \begin{matrix} s < 2 \\ n > 1/2 \end{matrix} \tag{19}$$

β is given as

$$\beta = \left(\frac{K}{A}\right)^{\frac{2-n}{3n}} (3nB)^{\frac{2}{3n}} \tag{20}$$

Introduction of this into the definition of ξ as given in Equation (11) gives a direct relation between ξ and y in terms of known quantities:

$$\xi = \left(\frac{K}{A}\right)^{\frac{2-n}{3n}} (3nBx)^{\frac{2}{3n}}\, y \tag{21}$$

The complete solution of the problem posed can now be given in terms of the velocity distribution u and v

$$u = (3nA^{n+1})^{-\frac{1}{3n}} \left[\frac{K^{n+1}}{Bx}\right]^{\frac{1}{3n}} F'$$

$$v = (3nx)^{\frac{1}{3n}-1} \left(\frac{K}{A}\right)^{\frac{2n-1}{3n}} B^{\frac{1}{3n}}\, [2\,\xi\, F' - F] \tag{22}$$

where A is defined by Equation (19) and F' by Equation (15). The maximum velocity in the middle of the jet can be immediately obtained from Equation (20) by inserting $F' = 1$.

Some typical velocity distributions are given in Figure 2 and 3. In Figure 2 F' (equal to u/u_o) is given as a function of ξ/A. In order to obtain a proper comparison

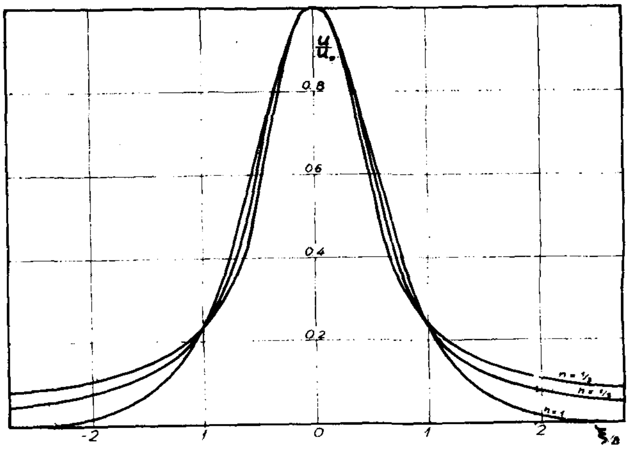

Fig. 2. The normalized axial velocity distribution for power-law exponents 1/3, 1/2, and 1.

* As Kapur's solution is given in an implicit form, the relations between momentum distance and velocity are hard to obtain. The solution obtained by Gutfinger is therefore given in order to facilitate the discussion of the properties of the jet and the limitations of the boundary-layer assumptions.

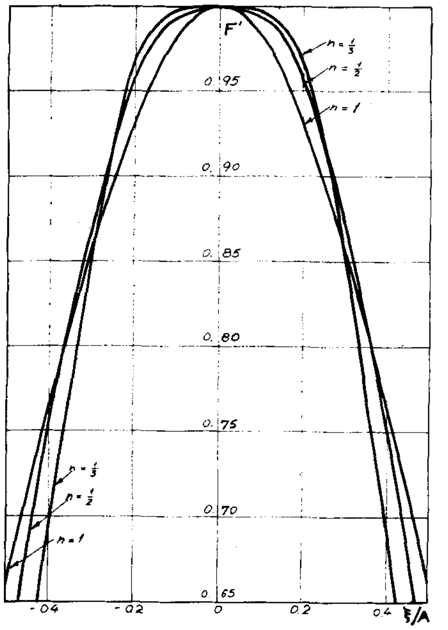

Fig. 3. Enlargement of the top portion of Figure 2.

the dimensionless coordinate ξ appearing in the function $F(\xi)$ was plotted with the scale factor $1/A$, thus making the value $\int_{-\infty}^{\infty} \frac{F'^2\,d\xi}{A}$ equal to unity. In this way the influence of u is more apparent as the comparison is made on the basis of equal momentum. Figure 3 is a magnification of the top part of Figure 2. It can be seen that near the maximum the distribution becomes flatter with decreasing n. This is similar to the flat portion near the maximum appearing in one-dimensional flow of such fluids. This property is general to symmetric free boundary layers in such fluids and is due to the fact that at the line of symmetry du/dy is zero. At a further distance (Figure 2) from the center the velocity gradient is steeper for smaller values of n. However the velocity does not go rapidly to zero as for the Newtonian case; instead the profile flattens at a nonnegligible value of F'. The latter feature is common to all boundary layers in power-law fluids and applies just as well to a flat plate as to a jet. In this sense boundary-layer assumptions in pseudoplastic fluids (power-law fluids with $n < 1$) are always less accurate than in the Newtonian fluids and the error increases with decreasing n. This is an inherent weakness of the power-law assumption itself, as for very small shear rates the apparent viscosity of this model approaches infinity. For actual fluids Equation (3) does not hold for small shear rates.

Two other parameters which are important to the understanding of the two-dimensional jet are the boundary-layer thickness δ and the entrainment Q. δ will be defined as the value of y corresponding to a velocity $u = 0.01\,u_0$; therefore

$$\delta = \left[\frac{K}{A}\right]^{\frac{n-2}{3n}} (3nBx)^{\frac{2}{3n}}\,\xi \text{ at } F' = 0.01 \qquad (23)$$

where ξ can be calculated from Equations (15) and (16). The total entrainment is defined as

$$Q_t = \int_{-\infty}^{\infty} u\,dy = 2\left(\frac{K}{A}\right)^{\frac{2n-1}{3n}} (3nBx)^{\frac{1}{3n}} \int_0^{\infty} F'\,d\xi = 2\frac{K}{A}^{\frac{2n-1}{3n}} (3nBx)^{\frac{1}{3n}} F(\infty) \qquad (24)$$

As $F(\infty) = \infty$ for $n \leq ½$ [see Equation (16)], the total entrainment for this case is infinitely large no matter what the other parameters are. This is due to the fact that the entrainment is unduly influenced by the value of n outside the boundary layer, where the authors' solution has no physical meaning. The very slow movement of the fluid outside the boundary layer is of no practical interest. To investigate the entrainment as a mixing operation it is much more useful to consider the entrainment Q in the boundary layer only. The entrainment defined in this way is the actual amount of fluid carried by the jet. This can be written as

$$Q = 2\left(\frac{K}{A}\right)^{\frac{2n-1}{3n}} (3nBx)^{\frac{1}{3n}} F \text{ at } F' = 0.01 \qquad (25)$$

The value of F at $F' = 0.01$ can be considered as a dimensionless entrainment coefficient.

Figure 4 gives a plot of $\frac{Q}{(K^{2n-1}Bx)^{\frac{1}{3n}}}$ vs. n.

Now consider the physical meaning of the above solution and its value in studying mixing problems. A general difficulty in comparing Newtonian and non-Newtonian fluids is the fact that there is no unique reference parameter describing the properties of the fluid. The properties of non-Newtonian fluids described by Equation (3) are given by two constants m (or B) and n. The only direct way to compare them is varying n for constant B. Such a comparison however is always of limited value. It is therefore much more useful to compare the effects of parameter variations in each fluid. For this purpose a class of fluids is uniquely defined by n. From Equation (13) it is evident that for a pseudoplastic fluid the velocity decrease with distance from the origin is steeper than for a Newtonian one. The width of the jet as described by δ increases more rapidly with distance for smaller values of n.

These statements are demonstrated in Figure 5. The upper part of this figure represents actual velocity distributions for three fluids at a distance 1 cm. from the origin having a constant kinematic momentum of 1,000 sq. cm./sec.². The effective viscosities of the liquids were chosen in such a way that the center-line velocities at $x = 1$ cm. are equal. The lower part of the figure represents velocity profiles of these liquids at a distance $x = 3$ cm. These profiles confirm the conclusions given above.

The increase of the entrainment with distance is more pronounced for pseudoplastic fluids. The latter effect however is more than balanced by the first, as the velocities become very small for larger distances.

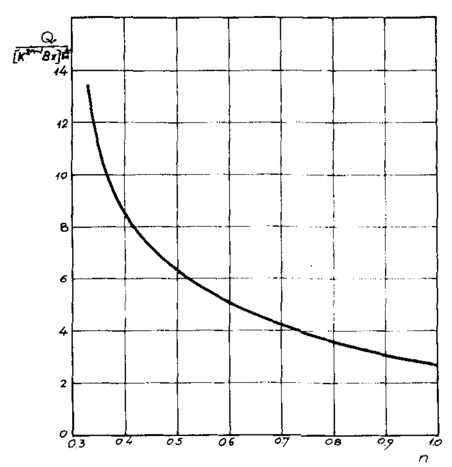

Fig. 4. The entrainment coefficient as a function of power-law exponent *n*.

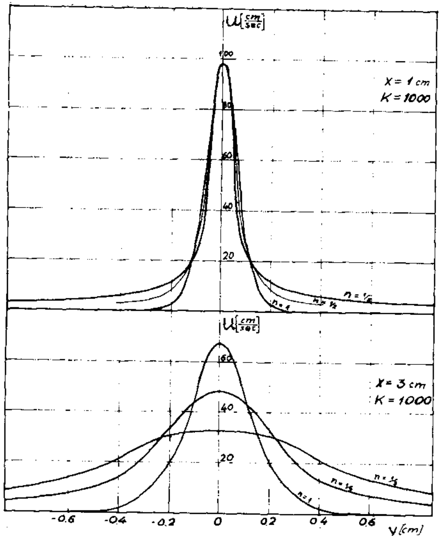

Fig. 5. Velocity profiles at distances $x = 1$ and $x = 3$ for liquids of power-law exponents 1/3, 1/2, and 1 with rheological constants B in units of 22.2, 5.3, and 0.1, respectively. The kinematic momentum is 1,000 cc./sec.^{2}.

The increase of the total momentum of the jet has a stronger effect on u and δ in the case of pseudoplastic fluids than in that of the Newtonian ones.

Of special interest is the effect of change of momentum on the entrainment. The value of Q increases with K for fluids with n larger than ½. For $n < ½$ the entrainment decreases with the increase of momentum. Higher momentum leads of course to higher velocities in the center of the jet. But as the width of the jet decreases even faster, the total result is a decrease in the volume flow. The case of $n = ½$ when Q is independent of K is similar to that of a round jet in a Newtonian fluid (*13*). Analogous conclusions can be made for the case of $1 < n < 2$ (dilatant fluids). However this case has not been considered in detail since it has never been shown that Equation (3) is a good approximation for such fluids over large ranges of shear rates. The analytical solutions of course apply equally well to this case.

Although the results obtained above are strictly correct only for the case of a two-dimensional laminar jet, qualitative conclusions can be drawn with reasonable confidence for laminar mixing processes in general. It appears immediately that any mixing due to entrainment will be more confined in space for a pseudoplastic fluid than for a Newtonian one. The immediate region in the neighborhood of the jet or agitator will be mixed more rapidly in a pseudoplastic fluid than in a Newtonian one, but the entrainment in the regions far from the mixer will be much slower. In practice a relatively small mixer rotating at high speeds will mix a large amount of a Newtonian fluid in a reasonable time owing to viscous entrainment. In a pseudoplastic fluid with a small value of n the same agitator will cause intense mixing in its immediate neighborhood, but the other part of the vessel will remain practically stagnant. These predictions are in good agreement with the experimental observations of Metzner and Taylor (*6*).

This illustrates some of the dangers of using apparent viscosity for design purposes. For small clearances, as in some paddle and sigma agitators, the flow can be approximated by simple shear flow, and the results will be quite reasonable. However when clearances are not small as compared with the agitator and vessel dimension, apparent viscosity becomes meaningless, and the results obtained from its use can be very misleading.

VALIDITY OF BOUNDARY-LAYER ASSUMPTIONS

Having obtained a solution for u and v one can now check initial boundary-layer assumptions for consistency. The following assumptions were made while Equation (4) was transformed into (5):

$$\frac{\partial v}{\partial x} \Big/ \frac{\partial u}{\partial y} << 1 \tag{26}$$

$$\left(\frac{\partial u}{\partial x}\right)^2 \Big/ \left(\frac{\partial u}{\partial y}\right)^2 << 1 \tag{27}$$

$$\frac{\partial v}{\partial x} \Big/ \frac{\partial u}{\partial x} << 1 \tag{28}$$

$$\frac{\partial^2 u}{\partial x^2} \Big/ \frac{\partial^2 u}{\partial y^2} << 1 \tag{29}$$

$$\frac{\partial u}{\partial x}\frac{\partial}{\partial x}\left[\left(\frac{\partial u}{\partial y}\right)^2\right]^{\frac{n-1}{2}} \Big/ \frac{\partial u}{\partial y}\frac{\partial}{\partial y}\left[\left(\frac{\partial u}{\partial y}\right)^2\right]^{\frac{n-1}{2}} = \frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x \partial y} \Big/ \frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial y^2} << 1 \tag{30}$$

The magnitude of each of the above expressions is given by

$$\frac{\partial v}{\partial x} \Big/ \frac{\partial u}{\partial y} = \left[A^{2-n}(3n)^{2-\frac{3n}{2}}\right]^{\frac{2}{3n}} \left[\frac{K^{2-n}x^{3n-2}}{B^2}\right]^{-2/3n} \cdot \left[(2\xi F' - F) - \frac{1}{3n}(F + 4\xi^2 F'')\right] << 1 \tag{31}$$

$$\left(\frac{\partial u}{\partial x}\right)^2 \Big/ \left(\frac{\partial u}{\partial y}\right)^2 = (9A^{2-n}n^2)^{\frac{2}{3n}}\left[\frac{K^{2-n}x^{3n-2}}{B^2}\right]^{-\frac{2}{3n}} \left(\frac{F'}{F''} + 2\xi\right)^2 << 1 \tag{32}$$

$$\frac{\partial v}{\partial x} \Big/ \frac{\partial u}{\partial x} = (9A^{2-n}n^2)^{\frac{1}{3n}}\left[\frac{K^{2-n}x^{3n-2}}{B^2}\right]^{-\frac{1}{3n}} \frac{\frac{1}{3n}(F + 4\xi^2F'') - (2\xi F' - F)}{F' + 2\xi F''} << 1 \tag{33}$$

$$\frac{\partial^2 u}{\partial x^2} \Big/ \frac{\partial^2 u}{\partial y^2} = \left[A^{2-n}(3n)^{2-\frac{3n}{2}}\right]^{\frac{2}{3n}} \left[\frac{K^{2-n}x^{3n-2}}{B^2}\right]^{\frac{-2}{3n}} \left[\left(\frac{1}{3n}+1\right)\left(\frac{F'}{F'''} + 2\xi\frac{F''}{F'''}\right) + \frac{2}{3n}\left(3\xi\frac{F''}{F'''} + 2\xi^2\right)\right] << 1 \tag{34}$$

$$\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x \partial y} \Big/ \frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial y^2} = \left[A^{2-n}(3n)^{2-3n}\right]^{\frac{2}{3n}} \left[\frac{K^{2-n}x^{3n-2}}{B^2}\right]^{\frac{-2}{3n}} \left(\frac{2}{F'''} + \frac{2\xi}{F''}\right)(F' + 2\xi F'') << 1 \tag{35}$$

For the Newtonian case the assumptions made are summarized by Equations (28) and (29). These relations are expressed in terms of the variables as following:

$$\frac{\partial v}{\partial x} \Big/ \frac{\partial u}{\partial x} = 1.71 \left[\frac{Kx}{B^2}\right]^{-2/3} \left[\frac{2\,\xi^2 F'' - 3\,\xi F' + 2F}{F' + 2\xi F''}\right] \quad (36)$$

$$\frac{\partial^2 u}{\partial x^2} \Big/ \frac{\partial^2 u}{\partial y^2} = 1.47 \left[\frac{Kx}{B^2}\right]^{-2/3} \left[2\frac{F'}{F'''} + 7\xi\frac{F''}{F'''} + 2\xi^2\right] \quad (37)$$

The relations given by Equations (31) to (35) can be written in a general form

$$\phi(n) \left[\frac{K^{2-n}\,x^{3n-2}}{B^2}\right]^{-\frac{1}{n}} f(\xi) << 1 \quad (38)$$

where $\phi(n)$ is a finite number for every n, but $f(\xi)$ is a function of position or ξ. The conditions given above will depend upon the magnitude of the dimensionless group $\left[\frac{K^{2-n}x^{3n-2}}{B^2}\right]$, which is essentially a modified Reynolds number. Obviously in order to fulfill the boundary-layer assumptions $f(\xi)$ must remain finite for all values of ξ, which are within the boundary layer (the width of which can be specified by a value of u/u_o or F'; for example $F' = 0.01$). This is true as far as the Newtonian fluid is concerned. However for a pseudoplastic fluid $f(\xi)$ is infinite for $\xi = 0$. This is due to the fact that the velocity profile near the center line is very flat for small values of n. For a pseudoplastic fluid there exists therefore a region near the center line where the boundary-layer assumptions can not be fulfilled. (However by increasing the modified Reynolds number this region can be made as small as desired.) This results in a basic difference between the boundary-layer assumptions for Newtonian and pseudoplastic fluids which becomes more pronounced the smaller the value of n.

There is another difference in applicability of boundary-layer assumptions for these two types of fluids which is more general and applies to all cases using boundary layer solutions. For high values of ξ the value of the Reynolds number needed to justify the assumptions becomes very large. When n is close to 1, this has but little influence since u/u_o becomes negligible for relatively small values of ξ. For smaller values of n this effect severely limits the region for which boundary-layer assumptions may be applicable.

In the region of the high-velocity gradient the boundary-layer assumptions hold quite well, as can be seen from the following numerical example. Consider a case of $n = 1/3$ and $\xi = 1.697$ which corresponds to a relative velocity $u/u_o = 0.5$. When one specifies the ratio of the neglected terms to those retained in the flow equation as 0.1, the various boundary-layer assumptions will be fulfilled at the following modified Reynolds numbers:

$$\frac{\partial v}{\partial x} \Big/ \frac{\partial u}{\partial y} = 305\, N_{Re}{}^{*-2} \leq 0.1; \qquad N_{Re}{}^{*} \geq 55.2 \quad (39)$$

$$\left(\frac{\partial u}{\partial x}\right)^2 \Big/ \left(\frac{\partial u}{\partial y}\right)^2 = 177\, N_{Re}{}^{*-2} \leq 0.1; \qquad N_{Re}{}^{*} \geq 42.0 \quad (40)$$

$$\frac{\partial v}{\partial x} \Big/ \frac{\partial u}{\partial x} = 28.4\, N_{Re}{}^{*-1} \leq 0.1; \qquad N_{Re}{}^{*} \geq 284 \quad (41)$$

$$\frac{\partial^2 u}{\partial x^2} \Big/ \frac{\partial^2 u}{\partial y^2} = 344\, N_{Re}{}^{*-2} \leq 0.1; \qquad N_{Re}{}^{*} \geq 58.6 \quad (42)$$

and

$$\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x \partial y} \Big/ \frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial y^2} = 136\, N_{Re}{}^{*-2} \leq 0.1; \qquad N_{Re}{}^{*} \geq 36.9 \quad (43)$$

The same calculation carried out for the Newtonian fluid gives the following results:

$$\frac{\partial^2 u}{\partial x^2} \Big/ \frac{\partial^2 u}{\partial y^2} = 15.13\, N_{Re}{}^{*-2/3} \leq 0.1; \qquad N_{Re}{}^{*} \geq 1{,}870$$

and

$$\frac{\partial v}{\partial x} \Big/ \frac{\partial u}{\partial x} = 3.25\, N_{Re}{}^{*-1/3} \leq 0.1; \qquad N_{Re}{}^{*} \geq 34{,}300.$$

It can be seen that the most severe assumptions in both cases are due to neglecting the flow equation in the direction normal to the main flow (y), as represented by the condition $\frac{\partial v}{\partial x} \Big/ \frac{\partial u}{\partial x} << 1$. Compared with this the neglected terms which are due to non-Newtonian behavior are of much smaller magnitude.

In a practical case there is an upper limit on Re^* due to the onset of turbulence. For the Newtonian fluid the actual range in which boundary-layer assumptions apply in a more strict sense is therefore quite limited. For non-Newtonian fluids no reliable data for the onset of turbulence in boundary layers are available as yet. In this case however there is another practical limitation; most of the pseudoplastic fluids are highly viscous, making the achievement of high Re^* number impractical. In order to check the applicability of the solution given above to real fluids one also has to take into account that very few fluids can be described by Equation (3) with reasonable accuracy. In most cases if α_1 is not constant, at least some of the higher coefficients are not zero. It has been shown (*14*) that even if these coefficients are small, the neglected terms do not necessarily become small as $Re^* \to \infty$. As an example one may consider the case where α_2 and α_3 are not zero. If α_2 and α_3 are small and constant, the normal stresses in the equation in the y direction become high for high Reynolds numbers. This however does not necessarily make the boundary-layer assumptions invalid.

When a regular magnitude analysis is carried out on the terms containing α_2, and α_3, the boundary-layer flow equation in the x direction becomes

$$\rho\frac{Du}{Dt} = \rho B \frac{\partial}{\partial y}\left|\frac{\partial u}{\partial y}\right|^n + \frac{\partial}{\partial y}\left\{\alpha_2\left[u\frac{\partial^2 u}{\partial x \partial y} + v\frac{\partial^2 u}{\partial y^2} + 2\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}\right]\right\} + 2\frac{\partial}{\partial x}\left[\alpha_3\left(\frac{\partial u}{\partial y}\right)^2\right] \quad (44)$$

The magnitude of the terms containing α_2 and α_3 compared with $B\left|\frac{\partial u}{\partial y}\right|^n$ is determined in the case of the jets in terms of the dimensionless groups

$$S_{\alpha 2} = \frac{\rho}{\alpha_2}\left(\frac{Bx}{K^{1-n/2}}\right)^{4/3n}$$
$$S_{\alpha 3} = \frac{\rho}{\alpha_3}\left(\frac{Bx}{K^{1-n/2}}\right)^{4/3n} \quad (45)$$

These groups represent the ratio between viscous and

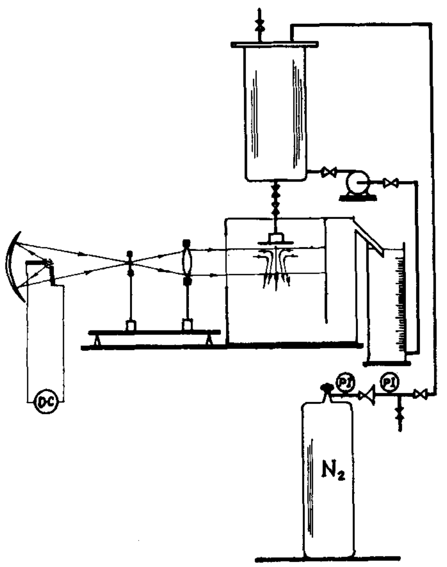

Fig. 6. Experimental apparatus.

(2). The two-dimensional jet was generated by passing the liquid through a converging nozzle which formed a slit 2 cm. long and 0.034 cm. wide at the exit. A baffle was placed at the exit from the nozzle, thus simulating the condition of a slit in a wall.

The liquid was forced through the nozzle from a pressure vessel connected through a reducing valve to a nitrogen cylinder. The liquid flowed into a rectangular tank built of Plexiglass and was held at constant height by means of an overflow. The momentum K was calculated from the measured total flow by assuming a flat velocity profile at the exit of the nozzle. The momentum of the jet could be changed by changing the pressure at the pressure vessel.

Aluminum particles (40 to 60μ) were introduced into the liquid, and the motion of these particles (and of the liquid) in the jet was photographed under illumination by a narrow parallel beam of light isolated from a carbon arc by means of a slit and condenser lenses. The narrow beam was focused on the jet in the direction of a plane normal to the nozzle. A 35-mm. camera, equipped with a 55-mm. f/1.8 lens and a set of extension bellows, was used. The photographic negatives which consisted of black streaks on a white background were superimposed under a photographic enlarger and traced on paper at 10X overall magnification. Only streaks with sharp ends were considered for measuring the velocity profiles.

elastic stresses. As $S_{\alpha 2}$ and $S_{\alpha 3}$ become very large, the terms containing α_2, α_3 become negligible, and Equation (7) applies. One notes that both Re^* and $S_{\alpha 2}$ (or $S_{\alpha 3}$) can go to infinity simultaneously only when $x \rightarrow \infty$, K and B remaining constant. If Re^* is made large by increasing K or decreasing B, this would tend to decrease S and thus increase the influence of the elastic terms. For a constant x the elastic terms should become large for high values of K, even though α_2/B is small. Similar results are obtained if α_2 and α_3 are directly proportional to α_1.

EXPERIMENTAL

Actual velocity profiles were obtained from experimental measurements for pseudoplastic as well as for Newtonian fluids. The experimental equipment used (Figure 6) and the method of measurement were similar to those described by Andrade

EXPERIMENTAL RESULTS, DISCUSSION

Figures 7 to 11 summarize the experimental results. Figure 7 gives a typical velocity distribution for a jet in a Newtonian liquid (80% glycerin solution in water having a viscosity of 74 centipoise at 16.8°C.). The curve represents the theoretical profile. In Figure 8 the center-line velocity u_o is plotted as a function of the distance from the origin of the jet for the above-mentioned conditions. For comparison the theoretical curve is given.

The pseudoplastic fluid used was a 0.5% aqueous solution of carboxy methyl cellulose (CMC). The flow properties of this solution (as obtained from a capillary viscosimeter) are described in Figure 9, where the outflow of the viscosimeter is plotted against the pressure gradient. The rheological constants m and n were calculated from this data by assuming that the properties of the fluid are described by Equation (3). (For a more detailed description of this method see references *8* and *16*.)

Figure 10 gives a typical velocity profile of a jet obtained in this solution. The theoretical curve plotted for comparison was calculated from Equation (22). (See Appendix.) Figure 11 gives the ratio $u_o/K^{\frac{n+1}{3n}}$ as a function of the distance from the origin x for two experiments.

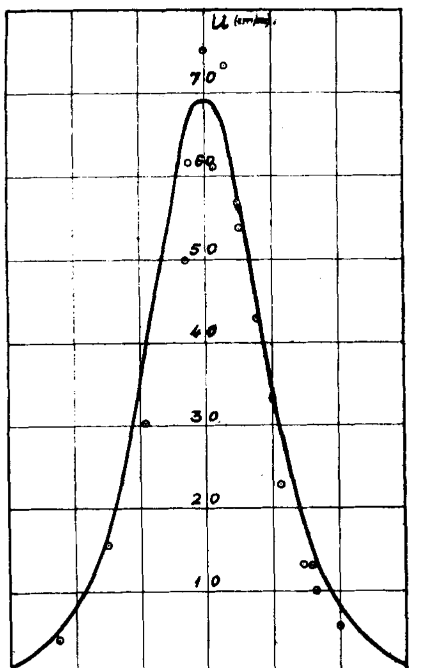

Fig. 7 Velocity profile at $x = 1$ cm. for a Newtonian jet. $\nu = 0.61$ stokes, $K = 1{,}460$ cc./sec.^{2}, temp. = 16.8°C.

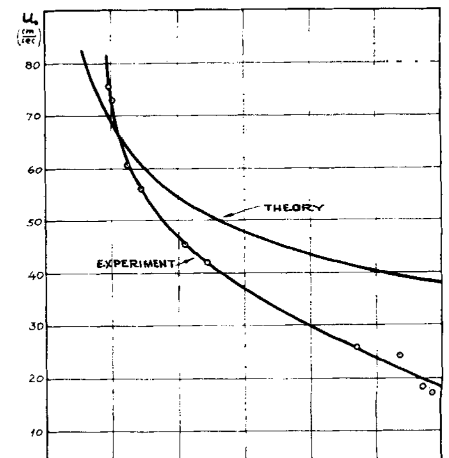

Fig. 8. Center-line velocity as a function of distance from the origin in a Newtonian jet.

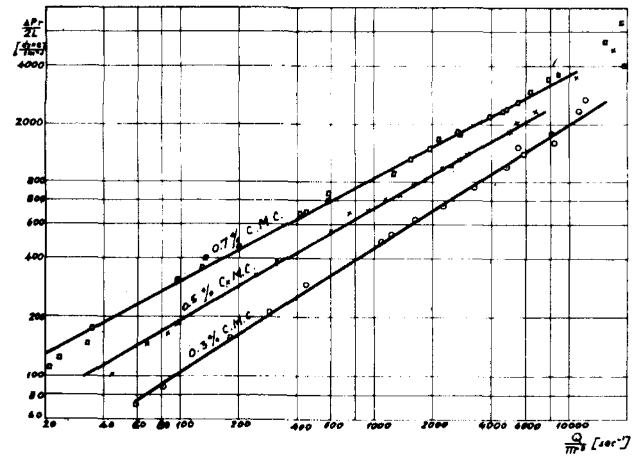

Fig. 9. Flow properties of aqueous solutions of carboxy-methyl-cellulose as obtained from capillary viscometer. The middle line (0.5% CMC) refers to the experiments described in this paper.

The theoretical curve was calculated from Equation (22).

In both the Newtonian and the non-Newtonian cases the agreement of the experiment with the theory is reasonable but not very good. For the Newtonian case this discrepancy has been explained by the difficulty of the experimental realization of the two-dimensional jet. The effect of the velocity distribution at the exit from the nozzle (small values of x) on the behavior of the jet is not negligible. Furthermore, owing to the finite length of the slit, the jet does not remain two dimensional for larger values of x. It is therefore surprising to note that the deviation of the results from the theory in the non-Newtonian fluid is of the same order of magnitude as in the Newtonian one. Seen in this light the results show a remarkably good agreement with the boundary-layer solutions obtained. Furthermore it has to be remembered that the fluid used in experiments, a CMC water solution, is known to have pronounced elastic properties. Another interesting observation was made during the experiments described above. In the non-Newtonian fluid, when K was decreased to 1,500 cc./sec.^{2}, no clearly defined jet could be observed. The liquid issuing from the nozzle spread immediately to the sides. This phenomenon was reproducible, and the transition was quite sharp. Similar effects could not be observed in the Newtonian fluid. Additional experiments are needed to determine if this is due to the power-law properties of the fluid or to elastic effects.

SUMMARY AND CONCLUSIONS

1. The boundary-layer equations of a two-dimensional jet in a pseudoplastic fluid are solved and discussed.

2. The validity of the boundary-layer assumptions implied in the formulation of the problem was investigated. The most severe limitation in both the Newtonian and the non-Newtonian case results from neglecting the flow equation in the direction normal to the main flow. For both types of fluids the relative magnitude of the neglected terms is uniquely described by a modified Reynolds number as defined in the paper. For the pseudoplastic fluid the boundary-layer assumptions do not apply on the center line of the jet. However the relative width of this section as compared with the total boundary-layer thickness decreases very rapidly for n exceeding 2/3. The general consistency of the boundary-layer assumptions with the solution doesn't differ from the Newtonian case for these fluids ($n > 2/3$).

For small values of n the thickness of the boundary layer is very strongly dependent on the assumption made on the velocity of the jet at the edge of the boundary layer, and the justification of the boundary-layer assumptions in this case is questionable. The influence of some higher-order terms in the flow equation of a general Rivlin Ericksen fluid was discussed. It was shown that even small values of the coefficients of the elastic terms could limit the range of validity of the boundary-layer assumptions.

3. Experimental measurements of velocity profiles in a two-dimensional jet for a power-law fluid (CMC solution) were carried out, and the results are in reasonable agreement with theory.

ACKNOWLEDGMENT

This paper is partly based on a M. Sc. Thesis submitted by Chaim Gutfinger to the Department of Chemical Engineering, Technion, Haifa, Israel. The authors are indebted to Dr. Zeev Rotem for many helpful suggestions and discussions and to Professor W. R. Schowalter for pointing out to us the work done by Kapur on the same subject.

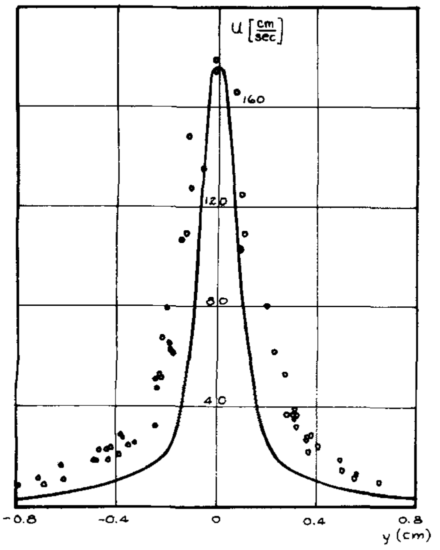

Fig. 10. Velocity profile at $x = 2$ cm. for a power-law fluid. $n = 0.57$, $B = 6.0$, $K = 4{,}600$ cc./sec.^{2}, temp. = 12°C.

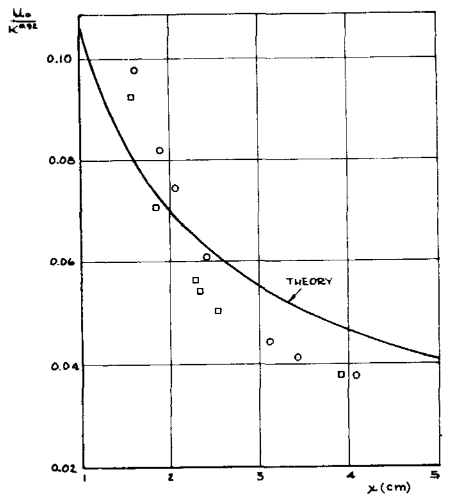

Fig. 11. $u_0/K^{\frac{n+1}{3n}}$ as a function of x for two experiments with a 0.5% CMC in water solution. □ — $K = 2{,}500$ cc./sec.^{2}, ○ — $K = 4{,}500$ cc./sec.^{2}.

NOTATION

a_o = dimensional constant, dimension $|T^2|$ magnitude unity
A = constant defined by Equation (18)
B = rheological constant
D_1 = kinematic matrix
D_2 = acceleration matrix
F = function of ξ
i, j = summation indices
K = kinematic momentum
M = momentum
m = rheological constant
n = rheological constant, exponent of power law
N_{Re}^* = modified Reynolds number
S_α = dimensionless group defined by Equation (45)
s = rheological constant
u = velocity in direction x
v = velocity in direction y
x, y, z = cartezian coordinates

Greek Letters

α_i = rheological constants
β = constant defined by Equation (20)
δ = boundary-layer thickness
ϵ = sign factor
μ = viscosity (Newtonian)
ν = kinematic viscosity (Newtonian)
ξ = independent variable of F
τ_{ij} = stress tensor
ψ = stream function

LITERATURE CITED

1. Acrivos, Andreas, et al., *A.I.Ch.E. Journal*, **6**, 312 (1960).
2. Andrade, E. N., *Proc. Phys. Soc. London*, **51**, 784 (1939).
3. Gutfinger, Chaim, M. Sc. thesis, Technion, Israel Institute of Technology, Haifa, Israel (June, 1962).
4. Jones, J. R., *Zeit Ang. Math Phys.*, **12**, 328 (1961).
5. Kapur, J. N., *J. Phys. Soc. Japan*, **17**, 1303 (1962).
6. Metzner, A. B., and J. S. Taylor, *A.I.Ch.E. Journal*, **6**, 109 (1960).
7. Rajeswari, G. K., and S. L. Rathna, *Zeit Ang. Math Phys.*, **13**, 43 (1962).
8. Reiner, M., "Deformation Strain and Flow," H. K. Lewis and Co., London (1960).
9. Rivlin, R. S., *J. Rat. Mech. Anal.*, **4**, 681 (1955).
10. ———, and J. L. Ericksen, *ibid.*, p. 323.
11. Rotem, Z., Ph. D. thesis, Technion, Israel Institute of Technology, Haifa, Israel (April, 1962).
12. Schlichting, Hermann, "Boundary Layer Theory," 4 ed., p. 166, McGraw-Hill, New York (1960).
13. *Ibid.*, p. 180.
14. Shinnar, R., to be published (1964).
15. Schowalter, W. R., *A.I.Ch.E. Journal*, **6**, 24 (1960).
16. Wilkinson, W. L., "Non-Newtonian Fluids," Pergamon Press, London (1960).

Manuscript received July 31, 1963; revision received February 4, 1964; paper accepted February 5, 1964. Paper presented at A.I.Ch.E. Houston meeting.

APPENDIX

Examples of Complete Analytical Solution of Boundary-Layer Equation for a Two-Dimensional Jet in a Power-Law Fluid

Some examples of complete analytical solution of the problem stated are tabulated below. The solution consists of integrating Equation (16), thus finding ξ as a function of F. Integrated Equation (16) and Equation (15) furnish the relations between ξ and F, thus enabling with the aid of (19), (21), and (22) to find the relation between u (and v) and y for given values of K, B, x, and n.

For comparison two cases resulting in a series expansion solution are included in Table 1.

TABLE 1.

n	s	ξ [Equation (16)]	ξ integrated	ξ [Equation (21)]	A	u [Equation (22)]
$\frac{1}{3}$	3	$\xi = \int \left[1 + \frac{1}{4}\lvert F\rvert^4\right] dF$	$\xi = F[1 + 0.05\lvert F\rvert^4]$	$\xi = 0.1485K^{5/3}(Bx)^{-2}y$		$u = 0.217K^{4/3}(Bx)^{-1}F'$
0.4	2.5	$\xi = \int \left[1 + \frac{1}{7}\lvert F\rvert^{7/2}\right]^2 dF$	$\xi = F\left[1 + \frac{4}{63}\lvert F\rvert^{3.5} + \frac{1}{392}\lvert F\rvert^7\right]$	$\xi = 0.334K^{4/3}(Bx)^{-5/3}y$	2.86	$u = 0.252K^{7/6}(Bx)^{-5/6}F'$
$\frac{1}{2}$	2	$\xi = \int \exp\left[\frac{1}{3}\lvert F\rvert^3\right] dF$	$\xi = F \sum_{k=0}^{\infty} \frac{1}{3k+1} \frac{\lvert F\rvert^{3k}}{n!\,3^k}$	$\xi = 0.226K(Bx)^{-4/3}y$	2.58	$u = 0.296K(Bx)^{-2/3}F'$
$< \frac{1}{2}$	>2	Equation (16)	$\xi = F\left[1 + \sum_{k=1}^{\infty} \frac{a(a+1)\ldots(a+k-1)}{(kb+1)k!} t^k\right]$ where: $a = \frac{1}{2-s}, b = s+1, t = \left\lvert\frac{F}{F(\infty)}\right\rvert^{s+1}$	Equation (21)	Equation (19)	Equation (22)